Common Notation

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Abstract

This note summarizes one comfortable notational practice. In most circumstances, this notation has been and still is convenient for me and many people I know. It turned out to be rather intuitive, mostly consistent, and generating relatively little ambiguity. I do not claim that the presented notation is convenient for everyone for every purpose.

1 Introduction

We are going to define a consistent system of symbols for everyday mathematical usage. This system has proved its practicability in the "formal methods" part of theoretical computer science, at least subjectively for the author and his environment. The system is composed of several standard, well-known notational solutions; it tries to avoid ambiguity and be uniform as far as possible.

We will be sloppy in defining the meaning of the symbols, assuming that for each concept, a reader either knows it already, or can look up its definition, or simply does not need the concept at all. Along with choosing symbols for the concepts, we will also take the freedom to choose definitions of concepts in some controversial cases (e.g., prefix/postfix points).

After that, we show interesting approaches to further disambiguation by using more letters.

2 Common Symbols and Terms

$\neg$, $\vee$, $\wedge$	Negation, disjunction, conjunction.
$\Rightarrow$	Implication. Sometimes longer: $\Longrightarrow$.
$\Leftrightarrow$	Equivalence. Sometimes longer: $\Longleftrightarrow$.
$\forall\, x{\in}X\colon \varphi$	For all members x of the set X the formula φ holds.
$\exists\, x{\in}X\colon \varphi$	There is a member x of the set X such that φ holds.
$\exists!$	Exists exactly one.
$\subseteq$	Left inclusion.

$\subsetneq$	Strict left inclusion.
$\subset$	Ambiguous. Avoid using it by default.
$\dot{\cup}$	Disjoint union. Example: $A \mathbin{\dot{\cup}} B$.
$\emptyset$	The empty set.
$\{\ \mid\ \}$	Class term. The term $\{x \mid \varphi\}$ denotes the class of all x that satisfy a formula φ. Example: $\{A \mid A \subseteq \{1, 2, 3\}\}$.
:=	Assignment statement in programs. Example: x:=5.
$\overset{\text{def}}{\iff}$	Defining-equivalence sign. Used to define the relation on the left-hand side via the formula on the right-hand side. Example: $A{\subsetneq}B \overset{\text{def}}{\iff} A{\subseteq}B \wedge \exists\, x{\in}B\colon x{\notin}A$.
$\overset{\text{def}}{=\!=}$	Defining-equality sign. Used to define the symbol on the left-hand side by the construct on the right-hand side. Example: $B \overset{\text{def}}{=\!=} \{A \mid A \subseteq \{1, 2, 3\}\}$.
id	For a set X, the term id_X denotes the binary identity relation on X, i.e., $\mathrm{id}_X \overset{\text{def}}{=\!=} \{(x, x) \mid x \in X\}$.
$\{\ ,\ \}_{\mathrm{K}}$	The term $\{x, y\}_{\mathrm{K}}$ is the Kuratowski pair $\{\{x\}, \{x, y\}\}$.
$(\ ,\ )$	The term (x, y) is the ordered pair of x and y without specifying how we define pairs.
$(\ ,\ ,\ )$	The term (x, y, z) is the ordered triple of x, y, and z without specifying how we define triples.
$\vdots$	
false	Boolean falsehood value.
true	Boolean veracity value.
$\mathbb{B}$	$\overset{\text{def}}{=\!=} \{\mathrm{false}, \mathrm{true}\}$.
$\mathbb{N}_+$	Natural numbers without zero.
$\mathbb{N}_{\geq 0}$	Natural numbers with zero.
$\mathbb{N}$	Ambiguous. Avoid using it by default.
$\mathbb{N}_n$	$\overset{\text{def}}{=\!=} \{x \in \mathbb{N}_+ \mid x \leq n\}$ for $n \in \mathbb{N}_{\geq 0}$.
$\mathbb{Z}$	Integers.
$\mathbb{Q}$	Rationals.
$\mathbb{Q}_+$	Positive rationals.
$\mathbb{Q}_{\geq 0}$	Nonnegative rationals.
$\mathbb{R}$	Reals.
$\mathbb{R}_+$	Positive reals.
$\mathbb{R}_{\geq 0}$	Nonnegative reals.
$\mathbb{C}$	Complex numbers.

$[\ ,\ ]$	Closed interval. The term $[a, b]$ denotes $\{x \mid a \leq x \leq b\}$ in a poset taken from the context. Typically, the poset is the set of real numbers with the usual ordering. The same choice of the poset applies to the next three forms.
$]\ ,\ ]$	Left-open and right-closed interval. The term $]a, b]$ denotes $\{x \mid a < x \leq b\}$.
$[\ ,\ [$	Left-closed and right-open interval. The term $[a, b[$ denotes $\{x \mid a \leq x < b\}$.
$]\ ,\ [$	Open interval. The term $]a, b[$ denotes $\{x \mid a < x < b\}$.
$\hookrightarrow$	The term $X \hookrightarrow Y$ denotes the set of injective (i.e., one-to-one) functions from X to Y.
$\twoheadrightarrow$	The term $X \twoheadrightarrow Y$ denotes the set of surjective (i.e., onto) functions from X to Y.
$\hookrightarrow\!\!\!\!\!\twoheadrightarrow$	The term $X \hookrightarrow\!\!\!\!\!\twoheadrightarrow Y$ denotes the set of bijective functions (i.e., one-to-one correspondences) from X to Y.
$\dashrightarrow$	The term $X \dashrightarrow Y$ denotes the set of partial functions from X to Y.
dom	Domain of a (potentially partial) function. Example: $\operatorname{dom} f$.
img	Image of a function (its set of values). Example: $\operatorname{img} f$.
gfp	The greatest fixpoint of a function (if exists). Example: $\operatorname{gfp} f$.
lfp	The least fixpoint of a function (if exists). Example: $\operatorname{lfp} f$.
prefp	The set of prefix points of a map on a poset. Defined as $\operatorname{prefp} f = \{x \in \operatorname{dom} f \mid f(x) \leq x\}$.
postfp	The set of postfix points of a map on a poset. Defined as $\operatorname{postfp} f = \{x \in \operatorname{dom} f \mid x \leq f(x)\}$.
mod	Ambiguous. First, it's an annotation of a modulo equivalence class, e.g., $5 \equiv 2 \pmod 3$. Second, it's a remainder after division, e.g., $5 \bmod 3 = 2$.
$\cdot\vert_\cdot$	Restricting a function's domain. Example: $f\vert_{\{4,5,6\}}$.
$\cdot \mid \cdot$	Divisible by. Example: $4 \mid 2$, but $4 \nmid 3$.
$\circ$	Right function composition. Example: $f \circ g$ means "f after g".
$⨾$	Left function composition. Example: $f ⨾ g$ means "f before g".
$\cdot^{⨾\cdot}$	Repeated function application. Example: $h^{⨾3} = h ⨾ h ⨾ h$.
$\mathcal{O}(\ )$	Landau Big O symbol. Example: $\mathcal{O}(n^2)$.
$\omega(\ )$	The term $\omega(f)$ denotes the set of maps asymptotically dominating a map f.
$\mathfrak{P}(\ )$	The term $\mathfrak{P}(X)$ denotes the power set of X.
$\mathfrak{P}_+(\ )$	The term $\mathfrak{P}_+(X)$ denotes the set of nonempty subsets of X.

$\cdot^{\cdot}$	Inherently ambiguous power notation. For a set X and an ordinal (or a nonnegative integer) n, the term X^n denotes the set of vectors (in other terms, sequences, words) of length n over X. If n is viewed as an ordinal, $X^n = (n{\to}X)$. For reals a and b, the term a^b denotes the arithmetic exponentiation, e.g., $5^3 = 125$. For a square matrix M and a natural number n, the term M^n means the multiplication of M with itself n times.
$\cdot^{\cdot\times\cdot}$	For a set X, the term $X^{m\times n}$ denotes the set of matrices over X with m rows and n columns.
$\cdot^{+}$	The term R^+ denotes the transitive closure of a binary relation R.
$\cdot^{*}$	The term R^* denotes the reflexive-transitive closure of a binary relation R.
ω	The smallest infinite ordinal.
$\cdot^{<\omega}$	The term $\cdot^{<\omega}$ denotes the set of finite words over an alphabet X.
$\cdot^{\omega}$	The term X^ω denotes the set of countably infinite words over an alphabet X.
$\cdot^{\leq\omega}$	The term $X^{\leq\omega}$ denotes the set of streams (i.e., finite and countably infinite words) over X.

An aside: Though typically the set of finite words over an alphabet X is denoted by X^* and the set of finite nonempty words by X^+, we are not going to use this notation because of clashes with reflexive-transitive and transitive closures.

Other required notation should be introduced on need. For instance, R^* may also mean the set of units of a ring R; writing L^* may mean the dual space of a vector space L, etc. It is acceptable to overload star in the right superscript position. But, if the reader does so, the contexts must clearly determine the current meaning.

Similarly, terms such as R^+, $[x]$, $\langle a\rangle$, $x{:}y$, etc. can also be ambiguous. They are often overloaded, so they should be defined on need, and their meanings should be clear from the context.

3 Alphabets

Sometimes the Latin alphabet is not sufficient to give names to variables and constants. Therefore, feel free to use the Greek, Hebrew, and Cyrillic alphabets. But beware that certain concepts are bound to more-or-less fixed symbols, e.g., lambda terms, the ratio of the circumference of a circle to its diameter, the cardinals, the shuffle operator, Galois connections, upper closures, etc.

3.1 Greek

Capital: Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ Ν Ξ Ο Π Ρ Σ Τ Υ Φ Χ Ψ Ω.
Small: α β γ δ ε/ϵ ζ η θ/ϑ ι κ λ μ ν ξ ο π/ϖ ρ/ϱ σ/ς τ υ ϕ/φ χ ψ ω.

3.2 Hebrew

א	ב	ג	ד
Aleph	Beth	Gimel	Daleth

Other Hebrew letters are difficult to get right by default in [PDF|Xe|Lua]LaTeX.

3.3 Cyrillic

Capital: А Б В Г Д Е Ё Ж З И Й К Л М Н О П Р С Т У Ф Х Ц Ч Ш Щ Ъ Ы Ь Э Ю Я.
Small: а б в г д е ё ж з и й к л м н о п р с т у ф х ц ч ш щ ъ ы ь э ю я.